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A GYROSCOPIC STABILIZATION PROPERTY OF A CONTROLLED CONSERVATIVE MECHANICAL SYSTEM

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ABSTRACT

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A stabilization property ("stabilizability") acquired by a not completely controlled conservative mechanical system with the application of additional gyroscopic forces is investigated. It is shown that such gyroscopic stabilization has features in common with the well known properties of the gyroscopic stability investigated by N. G. Chetayev (ref. 1). It is established, in particular, that an intrinsically unstable conservative system subjected to the gyroscopic effect can only acquire the property of stabilization by a control force of general nature, whereas control by a dissipative force necessarily upsets the stability in this case.

The effect discussed in this article can be used for the regulation of a mechanical system with a large number of degrees of freedom, when the initial construction of the controlled object does not provide sufficiently complete transference from the control units to all of the regulated components. The execution of this transference by the introduction of gyroscopic forces due to rotating masses is possibly more suitable than trying to accomplish the same end by the application of new coupling links (feedback or deedforward), because the introduction of

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gyroscopic forces does not elicit an additional penalty in terms of power expended in the motions of the object. Author

1. Consider a holonomic conservative (ref. 2, p. 59) mechanical system with n degrees of freedom, subjected to auxiliary controlling action and described by the following Lagrangian equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial \Pi}{\partial q_i}=b_i u \quad (i=1,\ldots,n). \tag{1.1}$$

Here q_i are the generalized coordinates, T is the kinetic, Π the potential energy; u is a scalar characterizing the magnitude of the control action, $b_i(q_1\ldots, q_n)$ are functions defining the direction of the force u. The functions T, Π, and b_i are given. We wish to find the control law $u = u(q_1\ldots, q_n, q_1'\ldots, q_n')$.

Let us suppose that for $u \equiv 0$ the system (1.1) has the equilibrium position $q_1 = 0$ $(i = 1\ldots, n)$. The equilibrium of a conservative mechanical system cannot be asymptotically stable in the Lyapunov sense (refs. 1 and 3). The problem consists in choosing the equation for $u(q,q')$ such that the equilibrium position becomes asymptotically stable. The process of choosing u so as to upgrade equilibrium to asymptotic stability will be called "stabilization" of the system. A system will be called "stabilizable" if its stabilization is possible.

We will investigate first-approximation equations (ref. 1 and 3) for the system (1.1) in the vicinity of the point $q_i = 0$, $q_i' = 0$:

$$\frac{d}{dt}\left(\frac{\partial T^\circ}{\partial q_i'}\right)+\frac{\partial \Pi^\circ}{\partial q_i}=b_i^\circ u \quad (i=1,\ldots,n). \tag{1.2}$$

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Here

$$T^\circ=\sum_{i,j=1}^{n} a_{ij}^\circ q_i' q_j', \quad \Pi^\circ=\sum_{i,j=1}^{n} b_{ij}^\circ q_i q_j, \quad a_{ij}^\circ, b_{ij}^\circ, b_i^\circ$$

are constants. We will say that the control

$$u = u^\circ(q, q') = \sum_{i=1}^{n} (p_i^\circ q_i + r_i^\circ q_i') \qquad (p_i^\circ = \text{const}, r_i^\circ = \text{const}), \tag{1.3}$$

found for the equations (1.2) stabilizes the system (1.1) in the first approximation if the equilibrium $q_i = q_i' = 0$ (i = 1..., n) of the system (1.1) is asymptotically stable for $u = u^\circ + \mu(q, q')$, even if there are nonlinear terms $T - T^\circ$, $\Pi - \Pi^\circ$, $\mu = u - u^\circ$.

Let us assume that the system (1.2) is completely controllable (ref. 4) by the action u. This means that for any initial conditions q_{i0}, q_{i0}' and time interval $0 \leqslant t \leqslant \tau$ there exists a control u(t) ($0 \leqslant t \leqslant \tau$) that will take the system (1.2) from the state $q_i(0) = q_{i0}$, $q_i'(0) = q_{i0}'$ to the state $q_i(\tau) = q_i'(\tau) = 0$ (i = 1..., n). From the complete controllability of the system (1.2) follows (ref. 5) the controllability (at the point $q_i = q_i' = 0$) also of the complete system (1.1) for all initial conditions q_{i0}, q_{i0}' from the neighborhood of the equilibrium position. Now the system (1.1) can (ref. 6) be stabilized in the first approximation by the control (1.3). Conversely, if the the system (1.2) is not completely controllable by the force u, then the system (1.1) is not stabilizable in the first approximation (ref. 6).

If the conservative system (1.1) in the first approximation (1.2) with $u \equiv 0$ is stable, then the conditions for complete controllability (ref. 7) coincide with the conditions (ref. 8) for which it is possible to stabilize the system (1.1) by a dissipative force with partial dissipation

$$b_i u = -\partial R / \partial q_i' \qquad (i = 1, \ldots, n), \tag{1.4}$$

where

$$R = \frac{1}{2}\left(\sum_{i=1}^{n} b_i q_i'\right)^2$$

is a positive Rayleigh function (ref. 2, p. 63). The following assertion is therefore valid.

A controlled conservative system (1.1) which is stable in the linear approximation (1.2) (for $u \equiv 0$) is stabilized in the first approximation by a force u of the class of forces (1.3) of a general nature when, and only when, it is so stabilized by a dissipative force

$$b_i^\circ u^\circ = -\frac{\partial R^\circ}{\partial q_i'} \quad (i = 1, \ldots, n), \tag{1.5}$$

where

$$R^\circ = \frac{1}{2}\left(\sum_{i=1}^{n} b_i^\circ q_i'\right)^2.$$

We now assume that the system (1.2) is not completely controllable by the force u, hence that the system (1.1) cannot be stabilized in the first approximation. Then, however, it is sometimes possible to achieve complete controllability of the first approximation system by the action u if we apply additional gyroscopic forces to the system (1.1). /158

The question of how to compute the gyroscopic forces so as to ensure this effect, as well as the derivation of effective criteria by which this calculation will be possible will not be considered herein. A special paper will be devoted to this problem. We will suffice to note the following. According to general control theory (ref. 4), the indicated mathematical problem will be solved if an antisymmetric matrix $\| g_{ij}^\circ \|$ is found, describing the gyroscopic forces in the linear approximation (ref. 2) such that, with the addition of the terms $g_{ij}^\circ q_j'$ to equations (1.2), the first approximation system will satisfy the general positional requirements (ref. 4). This can in fact be done in a broad class of situations. The only exceptions are systems having first integrals for

any choice of u and g_{ij}°, for example, systems (1.2) with two zero eigenvalues. The simplest illustrative examples, when the system becomes stabilizable due to the gyroscopic effect, will be considered below. Stabilizability that occurs as the result of gyroscopic forces will be called gyroscopic stabilizability.

Gyroscopic stabilizability is different from the stabilizability of a conservative system, just as the stability of a conservative system differs (ref. 1, pp. 102-112) from gyroscopic stability. It is the aim of the present article to investigate the properties of gyroscopic stabilizability.

2. Let the system (1.2) become completely controllable by the force u with the application of the gyroscopic effect. We write equations (1.2) in normal coordinates (ref. 1, pp. 97-99):

$$\frac{dx_{2i-1}}{dt} = x_{2i}, \quad \frac{dx_{2i}}{dt} = -\lambda_i x_{2i} + e_i^{\circ} u \quad (i = 1, \ldots, n), \tag{2.1}$$

where the vector $\{e_i^{\circ}\}$ is related to the vector $\{b_i^{\circ}\}$ by a nonsingular linear transformation, x_{2i-1} are the new coordinates, x_{2i} the velocities. The gyroscopic forces Q_i are described (ref. 2, p. 61) in the linear approximation Q_i° by the antisymmetric matrix $\{g_{ij}^{\circ}\}$ ($g_{ij}^{\circ} = -g_{ji}^{\circ}$; the g_{ij}° are constants):

$$Q_i^{\circ} = \sum_{j=1} g_{ij}^{\circ} x_{2j}.$$

Consequently, equations (2.1) assume the following form after application of the gyroscopic forces:

$$\frac{dx_{2i-1}}{dt} = x_{2i}, \quad \frac{dx_{2i}}{dt} = -\lambda_i x_{2i-1} + \sum_{j=1}^{n} g_{ij}^{\circ} x_{2j} + e_i^{\circ} u \tag{2.2}$$
$$(i = 1, \ldots, n).$$

We will confine the ensuing discussion to cases when there are no zeros among the eigenvalues λ_i. Then the following assertion is valid.

Lemma 2.1. If, with the application of gyroscopic forces to the conservative system (1.1), the linear approximation system (2.2) turns out to be completely controllable by the force u, then this system (2.2) will also be completely observable (ref. 4) in terms of the quantity

$$\gamma(t) = \sum_{i=1}^{n} e_i^{\circ} x_{2i}(t). \tag{2.3}$$

Note 2.1. The system (2.2) is called completely observable in terms of /159 quantity (2.3) if, for the interval $0 \leqslant t \leqslant \tau$ and for any coordinate x_k ($k = 1..., 2n$), it is possible to indicate functions $v_{k\gamma}(t)$ and $v_{ku}(t)$ which will renew the value of $x_k(\tau)$ according to observation of the functions $\gamma(t)$ and $u(t)$ ($0 \leqslant t \leqslant \tau$), by means of the linear operation

$$x_k(\tau) = \int_0^{\tau} [\gamma(t) v_{k\gamma}(t) + u(t) v_{ku}(t)]\, dt. \tag{2.4}$$

In reference 4 a definition of complete observability is given for a free linear system ($u \equiv 0$). Taking the control u(t) into account does not substantially alter the problem of observation (ref. 9). The fact that the property of observability of a system does not depend on the type of control or the method by which it is introduced into the system has been noted by R. E. Kalman.

The duality principle (ref. 4) between control and observation can be used to prove lemma 2.1. According to this principle, the system

$$\frac{dz}{dt} = Az + bu$$

is completely controllable by the force u when, and only when, the conjugate system

$$\frac{dz}{dt} = -A_z^{*}.$$

is completely observable according to the quantity $\gamma = b^*z$. Here z, b are vectors, A is a matrix, and the asterisk (*) denotes the transpose. Consequently, to prove the lemma it is sufficient to show that in the given case the conjugate system

$$\left.\begin{aligned} \frac{dy_{2i-1}}{dt} &= \lambda_i y_{2i}, \\ \frac{dy_{2i}}{dt} &= -y_{2i-1} - \sum_{j=1}^{n} g_{ij}^{\circ} y_{2j} + e_i^{\circ} u \quad (i = 1, \ldots, n) \end{aligned}\right| \tag{2.5}$$

is completely controllable by the action u.

Carrying out the nonsingular transformation

$$y_{2i-1} = -\lambda_i x_{2i-1}, \; y_{2i} = x_{2i} \tag{2.6}$$

and taking into account the antisymmetry of the matrix $\{g_{ij}^{\circ}\}$, we return from the system (2.6) to the system (2.2). The system (2.2) is completely controllable by convention. The transformation (2.6) does not violate the property of complete controllability, so that the system (2.5) is also completely controllable. Hence, according to the duality principle, the system (2.2) is completely observable in terms of the quantity (2.3). This proves the lemma.

We will discern three cases below.

Case 2.1. The conservative system (2.1) with $u \equiv 0$ is stable.

Case 2.2. The system (2.1) is unstable and remains unstable with the application of gyroscopic forces (for $u \equiv 0$).

Case 2.3. The system (2.1) is unstable but with the application of gyroscopic forces becomes stable (nonasymptotically, ref. 1, pp. 106-108) (for $u \equiv 0$).

3. Let us analyze the first case. Inasmuch as the application of gyroscopic forces causes the system (2.1) to go over to the completely controllable

system (2.3), stabilization of the system in the first approximation becomes possible. An essential fact is that the gyroscopic stabilizability thus generated preserves, in the first case, the property of stabilizability, noted in section 1, of a stable conservative system. Namely, the following assertion must be true.

Theorem 3.1. Let the system (1.2) be stable but not completely controllable by the force u. With the application of gyroscopic forces, the system (1.1) becomes stabilizable in the first approximation by proper choice of control u in the class of forces (1.3) of a general nature when, and only when, it becomes so stabilizable by a dissipative force.

Proof. For proof of the theorem it is sufficient to show that the /160
completely controllable system (2.2) is stabilized by the dissipative force (1.5), which is expressed in terms of the coordinates x_k as follows:

$$e_i^\circ u = -\frac{\partial R^\circ}{\partial x_{2i}}, \qquad R^\circ = \frac{1}{2}\left(\sum_{i=1}^{n} e_i^\circ x_{2i}\right)^2 \qquad (i = 1, \ldots, n). \tag{3.1}$$

For this, we inspect the function

$$V(x) = \frac{1}{2}\sum_{i=1}^{n} (\lambda_i x_{2i-1}^2 + x_{2i}^2), \tag{3.2}$$

which is equal to the total energy of the system (2.1). The function V(x) satisfies all of the conditions of the theorem (ref. 10 and ref. 11, p. 81) on asymptotic stability. In fact, the system (1.2) or, what amounts to the same thing, the system (2.1) is stable for $u \equiv 0$. Consequently, all numbers $\lambda_i > 0$. This means that the function V (3.2) is positive definite. Furthermore, the system (2.2) does not have nonequilibrium (differing from $x \equiv 0$) motions x(t) in which the negative derivative.

$$\frac{dV}{dt} = -2R^\circ = -\left(\sum_{i=1}^{n} e_i^\circ x_{2i}\right)^2 \tag{3.3}$$

would invert identically to zero by virtue of (2.2) with the control u (3.1). This fact is a consequence of the complete observability of the system (2.2) in terms of the quantity (2.3), as established in lemma 2.1. We in fact suppose the contrary to be true, that the system (2.2) with the control u (3.1) has a nonzero motion x(t), where $dV/dt \equiv 0$. In such motion also $u \equiv 0$. But this means that the free system (2.2) (i.e., for $u \equiv 0$) has a motion x(t) satisfying the conditions

$$\left.\begin{aligned} &\gamma(t) = \sum_{i=1}^{n} e_i^\circ x_{2i}(t) \equiv 0 \text{ при } 0 \leqslant t \leqslant \tau, \\ &x_k(\tau) \neq 0 \quad (k = 1 \text{ или } k = 2, \ldots, \text{ или } k = 2n), \end{aligned}\right| \tag{3.4}$$

where τ is some positive number. If such were actually the case, the quantity $x_k(\tau)$ could not possibly be renewed in terms of the quantity $\gamma(t)$ by means of the operation (2.4). The operation (2.4), however, is executed as the result of the complete observability of the system (2.2). The contradiction implies $dV/dt \not\equiv 0$ if $x(t) \not\equiv 0$. Thus, the function V(x) (3.2) satisfies all conditions of the aforesaid theorem (refs. 10 and 11). This proves theorem 3.1.

Note 3.1. The dissipative force u (3.1) (or, equivalently, the force (1.5)) can be interpreted as a solution to the problem (ref. 12) of analytically designing an optimum controller u(x) that stabilizes the linear approximation system (2.2) and, in so doing, minimized the quantity

$$J^\circ = \int_0^\infty [2R^\circ(x(t)) + u^2(t)]\,dt. \tag{3.5}$$

neighborhood of the lower stable equilibrium position and subjected to a control torque u acting solely on the coordinate ϕ (fig. 1).

The gyroscopic effect is produced in this case by the rotation of a pilot wheel m about the pendulum axis. The system is stabilized by the torque $u = -\phi$.

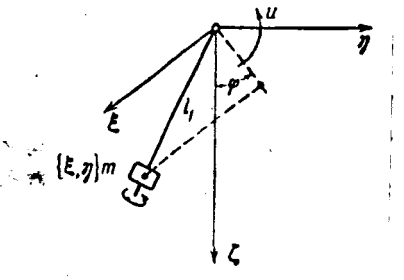

Figure 1.

4. We now examine the second case. Assume that, after application of gyroscopic forces, the system remains unstable but becomes completely controllable by the action u. Then, here again, it is possible to construct a control u (1.3) stabilizing the system (3.6) in the first approximation. However, it is important to recognize that now it is no longer possible in stabilization (ref. 1) to get by with just the dissipative force u (1.4); a force of a more general nature is needed. Essentially, this force u can be computed if we solve the problem of analytically designing an optimum controller (ref. 12) for the first approximation system (2.2). Here again, for the quantity to be minimized we can choose the integral J (3.5). However, the optimum positive definite Lyapunov function V^{o} obtained in solving the problem (3.5) will no longer have the sense of the total energy of the system (2.1).

Example 4.1. A simple example, in which the conditions enunciated in this section are realized, is a double pendulum with four degrees of freedom $(\xi_1, \eta_1, \xi_2, \eta_2)$ moving in a neighborhood of the upper position of unstable

Analogously, the force u (1.4) can be interpreted as a solution to the problem of analytically designing an optimum controller u(q,q′) that stabilizes/161 the complete system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial \Pi}{\partial q_i}=Q_i+b_iu \quad (i=1,\ldots,n) \tag{3.6}$$

and minimizes the following quantity in the motions of this system:

$$J=\int_0^\infty [2R(q'(t))+u^2(t)]\,dt. \tag{3.7}$$

Here the solution to the optimization problem (3.7) turns out to be possible because (refs. 13, and 14) the application of the gyroscopic forces Q_i to the system (1.1) renders this system completely controllable in the first approximation.

In order to be certain of the validity of the above assertions, we must formulate the Lyapunov-Bellman equations (ref. 13) for the problems (3.5) and (3.7). Then it happens that the optimum Lyapunov function V^o satisfying this equation is equal to twice the total energy of the system (1.2) or (1.1), respectively, and the optimum control is actually equal to the quantities (1.5) or (1.4); but the minimum values of the integrals (3.5) and (3.7) have the sense here of twice the amount of energy dissipated in the motions of the optimum systems. Therefore, the problem (3.7) for the system (3.6) is an example of the problem of analytically designing an optimum controller for a nonlinear system, when the solution can be determined from physical considerations, obviating the cumbersome computations required (refs. 13 and 14) in solving this type of problem in the general case.

Example 3.1. A simple example, in which the conditions of theorem 3.1 are realized, is a pendulum with two degrees of freedom (ξ, η) moving in a

equilibrium and subjected to a control torque acting on the coordinate ϕ (fig. 3). The gyroscopic effect induced by rotation of a pilot wheel m about the rod l_1 cannot produce stability of the system, but complete controllability of the latter can be attained, hence its stabilizability by the control $u = u(\xi_i, \xi_i', \eta_i, \eta_i')$ is also attainable.

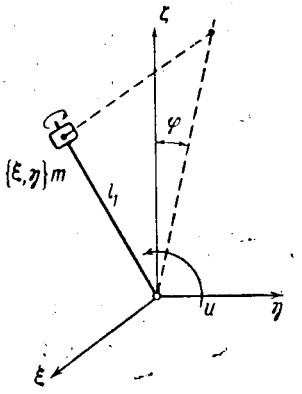

Figure 2.

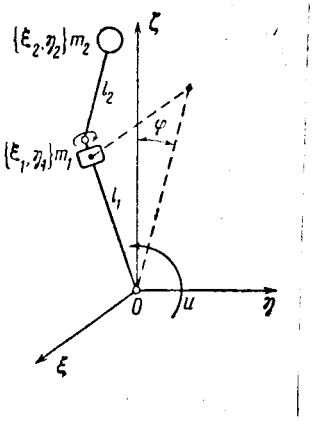

Figure 3.

Example 4.2. A second, simpler example of the situation analyzed in this section is a pendulum with two degrees of freedom (ξ, η) moving in a neighborhood of the upper position of unstable equilibrium and also controlled by the torque u acting on the coordinate ϕ (fig. 2). /162

The gyroscopic effect induced by slow rotation of the pilot wheel m is insufficient for rendering the system stable, but it does make it completely controllable by the function u. Let the system of first approximation equations (2.2) be written in this case in the form

$$x_1' = x_2, \quad x_2' = x_1 - \omega x_4, \quad x_3' = x_4, \quad x_4' = x_3 + \omega x_2 + u, \tag{4.1}$$

where x_1, x_3 represent the coordinates ξ and η, x_2, x_4 the velocities ξ' and η',

and ω is proportional to the velocity of rotation of the pilot wheel m about the rod l_1. For small values of ω, the system (r.1) is unstable (for u $\equiv$ 0). The stabilizing function, minimizing the quantity

$$J = \int_0^\infty [x_4^2(t) + u^2(t)]\, dt, \tag{4.2}$$

has the form

$$u = -\sum_{i=1}^{4} p_i^\circ x_i, \tag{4.3}$$

where

$$p_1^\circ = -\frac{2}{\omega}\sqrt{9 - 2\omega^2 + 2\sqrt{4 + (\omega^2 - 4)^2}}, \quad p_2^\circ = \omega - \frac{4}{\omega} - \frac{1}{\omega}\sqrt{4 + (\omega^2 - 4)^2},$$
$$p_3^\circ = 0, \quad p_4^\circ = \sqrt{9 - 2\omega^2 + 2\sqrt{4 + (\omega^2 - 4)^2}}. \tag{4.4}$$

5. We next consider the third case. After the application of gyroscopie forces, let the first approximation system become both completely controllable by the force u and stable. Then here, as in section 3, the system (2.2) is stable and stabilizable. However, this stabilizability differs signaficantly from the stabilizability investigated in section 3. Specifically, the stabilizability property comprising the heart of theorem 3.1 is no longer preserved here. On the contrary, the following assertion is valid.

Theorem 5.1. Let the conservative system (1.1) in the first approximation (1.2) be unstable and not completely controllable by the action u. We assume that with the application of suitable gyroscopic forces the system (1.2) will go over to the stable and completely controllable system (2.2). Then the system (3.6) is stabilized in the first approximation by the force (1.3) of the class of general forces. Now, however, not only can the system (3.6) not be stabilized by the dissipative force (1.4), but, to the contrary, the dissipative force

(1.4) with partial dissipation necessarily upsets the stability invested in /163
the system (3.6) for $u \equiv 0$.

Note 5.1. Theorem 5.1 states, therefore, that the dissipative force u (1.4) with partial dissipation necessarily violates the stability of the system (3.6), if this gyroscopically stable system can only be consolidated in the first approximation to asymptotic stability by the general type force u (1.3).

Proof of theorem 5.1. Consistent with all that has gone before, it is sufficient to show that the dissipative force (1.4) necessarily upsets the stability of the equilibrium position $q_i = 0$ of the system (3.6) if only with the application of the gyroscopic forces Q_i the unstable system (1.1) goes over to the completely controllable (in the first approximation) system (3.6). We will show this to be true.

The function $V = H(q,q')$, where H is the total energy of the system (1.1), is sign-alternating in the present case. Its derivative dV/dt in the direction of the motions of the system (3.6) with the control function u (1.4) satisfies the equation

$$\frac{dV}{dt} = -2R.$$

Consequently, the quantity dV/dt is a negative function and can revert identically to zero only for $R(q(t), q'(t)) \equiv 0$. According to lemma 2.1, the system (2.2) is completely observable in terms of the linear quantity $\gamma(t)$ (2.3), i.e., in terms of $\gamma(t) = R^{0}(q(t), q'(t))$. But, according to reference 15, this would be altogether impossible if there existed motions $\{q_i(t), q_i'(t)\}$ differing from the equilibrium position and existing such that in them $R = \gamma(t) \equiv 0$. Consequently, there are no motions $\{q_i(t), q_i'(t)\}$, different from $q_i = 0$, $q_i' = 0$ $(i = 1..., n)$, in which $dV/dt \equiv 0$. This means that the

function V in this case meets all the conditions of the instability theorem (ref. 11, p. 84). Consequently, the instability of the equilibrium position of the system (3.6) is established. This proves theorem 5.1.

Example 5.1. An example in which the conditions of theorem 5.1 are fulfilled is the pendulum investigated in example 4.2 under the condition of a sufficiently rapidly rotating pilot wheel m. The system (4.1) is stable for large values of ω (for $u \equiv 0$). This stability is upset by the dissipative force $u = -x_4$ (or $u = \phi'$). The force u that stabilizes the system (4.1) and, in so doing, minimizes the quantity (4.2) has, as in example r.2, the form (4.3), (4.4).

We note in conclusion that the results given in this article for controlled mechanical systems are intimately related to the investigations of N. G. Chetayev (ref. 1) and with several other investigations on the stability of mechanical systems (see, e.g., refs. 8 and 16 to 22). The present work exhibits attributes in common with the investigations just mentioned. Here, certain well known postulates take on a new interpretation from the viewpoint of control theory and the theory of analytical controller design.

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